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Factorising usco mappings

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We deal with factorisations through metrizable spaces of compact-valued u.s.c. mappings. In case the domain has some higher separation axioms, we found some natural relationship with the graph of such mappings. For an arbitrary domain, we related such factorisations to compact-valued continuous expansions.

1. Introduction

All spaces in this paper are assumed to be T_1-spaces. For a space Y, we will use 2^Y to denote the *power set* of Y, i.e. the set of all subsets of Y. Also, let

$$\mathscr{F}(Y) = \{S \in 2^Y\colon\ S \neq \varnothing \text{ and } S \text{ is closed}\}, \quad \text{and}$$
$$\mathscr{C}(Y) = \{S \in \mathscr{F}(Y)\colon\ S \text{ is compact}\}.$$

Any relation $R \subset X \times Y$ can be represented as a map $\Psi_R : X \to 2^Y$ by letting $\Psi_R(x) = \{y \in Y\colon\ (x, y) \in R\}$, $x \in X$. This map is usually called *set-valued*, or *multi-valued*, and sometimes a *multifunction*. The converse is also true. To any set-valued mapping $\Psi : X \to 2^Y$ one can associate the relation

$$\mathsf{Graph}(\Psi) = \{(x, y) \in X \times Y\colon\ y \in \Psi(x)\},$$

which is called the *graph* of Ψ. Thus, we have that $R = \mathsf{Graph}(\Psi_R)$. Motivated by this, for a property $\mathscr{P}$ of subsets of topological spaces, we will often say that $\Psi : X \to 2^Y$ has a *$\mathscr{P}$-graph* if $\mathsf{Graph}(\Psi)$ has the property $\mathscr{P}$ as a subset of $X \times Y$.

For a mapping $\Phi : X \to 2^Y$ and $B \subset Y$, let $\Phi^{-1}[B] = \{x \in X\colon\ \Phi(x) \cap B \neq \varnothing\}$. We say that $\Phi : X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set $\Phi^{-1}[U]$ is open in X for every open $U \subset Y$. A mapping $\Psi : X \to 2^Y$ is *upper semi-continuous*, or u.s.c., if the set

$$\Psi^{\#}[U] = X \setminus \Psi^{-1}[Y \setminus U] = \{x \in X\colon\ \Psi(x) \subset U\}$$

is open in X for every open $U \subset Y$. For convenience, we say that $\Psi : X \to 2^Y$ is *usco* if it is u.s.c. and nonempty-compact-valued.

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Let $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping, where Y is a metrizable space. A triple (Z, g, φ) is an *l.s.c. weak-factorisation* of Φ [2,20] if Z is a metrizable space with a topological weight $w(Z) \leqslant w(Y)$, $g : X \to Z$ is continuous, and $\varphi : Z \to \mathscr{F}(Y)$ is l.s.c. such that $\varphi(g(x)) \subset \Phi(x)$ for every $x \in X$. As remarked in [20], the term "factorisation" was reserved for the special case when $\varphi(g(x)) = \Phi(x)$ for all $x \in X$. In this paper, we deal with factorisations of usco mappings. Namely, we consider usco mappings $\Psi : X \to \mathscr{C}(Y)$, where Y is a metrizable space, and we are interested in conditions on Ψ and/or X under which there exists another metrizable space Z, with $w(Z) \leqslant w(Y)$, a continuous map $g : X \to Z$ and an usco mapping $\psi : Z \to \mathscr{C}(Y)$ such that $\Psi(x) = \psi(g(x))$ for all $x \in X$, i.e. $\Psi = \psi \circ g$. In this case, we say that the triple (Z, g, ψ) is an *usco factorisation* of Ψ. Some natural conditions for usco factorisations were investigated in [7], and some affirmative results were obtained for the case of a separable Y (see [7, Theorem 3.1]). Other expansion-type of conditions were obtained in [10, Lemma 4.7]. Here, we generalise all these results from a common point of view. Our approach is based on the fact that every usco mapping between metrizable spaces has a closed graph, hence it has a zero-graph (that is, its graph is a zero-set of $X \times Y$). This implies the following simple observation which verification is left to the reader.

Proposition 1.1. *If Y is a metrizable space and $\Psi : X \to \mathscr{C}(Y)$ has an usco factorisation (Z, g, ψ), then Ψ is an usco mapping with a zero-graph.*

We are now ready to state also the main contribution of this paper. Namely, here we demonstrate the converse of Proposition 1.1 in a very general setting (see Theorem 5.1), and illustrate particular cases by way of example (see Corollary 5.3). If the domain is paracompact, then "zero-graph" is further relaxed to "G_δ-graph", see Corollary 5.5. Several other results are obtained as well, see Section 5. The main ingredient of all these results is given by special "approximate" intermediate factorisations which are obtained in the next section. Section 3 provides the main interface to "pure" intermediate factorisations, while Section 4 is devoted to some supporting results for the so called strongly u.s.c. mappings. The things are finally culminating in Section 5, where the proof of Theorem 5.1 is accomplished. The last section 6 deals with the case when no extra conditions on the domain are assumed a priori, see Theorem 6.3.

2. Approximate usco weak-factorisations

For a cover $\mathscr{V}$ of Y and $A \subset Y$, we will write that $\mathrm{tb}(A) < \mathscr{V}$ if $A \subset \bigcup \mathscr{W}$ for some finite $\mathscr{W} \subset \mathscr{V}$. We use $\mathrm{st}(A, \mathscr{V})$ to denote the *star* of A with respect to $\mathscr{V}$, i.e. $\mathrm{st}(A, \mathscr{V}) = \bigcup\{V \in \mathscr{V}\colon\ V \cap A \neq \varnothing\}$. In particular, to every $\Phi : X \to 2^Y$ we associate another mapping $\mathrm{st}[\Phi, \mathscr{V}] : X \to 2^Y$ defined by $\mathrm{st}[\Phi, \mathscr{V}](x) = \mathrm{st}(\Phi(x), \mathscr{V})$, $x \in X$. Finally, we write $\mathrm{tb}(\Phi) < \mathscr{V}$ if $\mathrm{tb}(\Phi(x)) < \mathscr{V}$ for every $x \in X$.

For a metric space (Y, d), $y \in Y$ and $\varepsilon > 0$, let $B^d_\varepsilon(y) = \{z \in Y\colon\ d(y, z) < \varepsilon\}$ be the *open ε-ball* centred at y; and, for a subset $A \subset Y$, let

$$B^d_\varepsilon(A) = \bigl\{y \in Y\colon\ d(y, A) < \varepsilon\bigr\} = \bigcup\bigl\{B^d_\varepsilon(a)\colon\ a \in A\bigr\}.$$

A u.s.c. mapping $\Psi : X \to 2^Y$ is *strongly u.s.c.* if $\Psi^{-1}[F]$ is a zero-set of X for every zero-set F in Y, see [7]. A mapping $\Psi : X \to 2^Y$ has the *locally-finite lifting property* [8, (3.3)] (see, also, [9,18]) if for every locally-finite family $\mathscr{F}$ of closed subsets of Y, there is a locally-finite family $\{U_F\colon\ F \in \mathscr{F}\}$ of open subsets of X such that $\Psi^{-1}[F] \subset U_F$ for each $F \in \mathscr{F}$. A mapping $\Psi : X \to 2^Y$ is a *multi-selection* (or, a *set-valued selection*) for $\Phi : X \to 2^Y$ if $\Psi(x) \subset \Phi(x)$ for every $x \in X$. Finally, for a set $\mathscr{A}$, let $\ell_1(\mathscr{A})$ be the Banach space of all functions $s : \mathscr{A} \to \mathbb{R}$ such that $\sum_{\alpha\in\mathscr{A}} |s(\alpha)| < \infty$, where the linear operations are defined pointwise and the norm is $\|s\| = \sum_{\alpha\in\mathscr{A}} |s(\alpha)|$ for each $s \in \ell_1(\mathscr{A})$.

Theorem 2.1. *Let $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping, where X is a normal space and Y is paracompact. Then, for a u.s.c. multi-selection $\Psi : X \to \mathscr{F}(Y)$ of Φ the following are equivalent*:

(a) *Ψ has the locally-finite lifting property.*
(b) *For every open cover $\mathscr{V}$ of Y there exists a continuous map $g : X \to \ell_1(\mathscr{V})$ and a u.s.c. mapping $\theta : g(X) \to \mathscr{F}(Y)$ such that*

$$\mathrm{tb}(\theta) < \mathscr{V} \quad \textit{and} \quad \Psi(x) \subset \theta\bigl(g(x)\bigr) \subset \mathrm{st}[\Phi, \mathscr{V}](x) \quad \textit{for all } x \in X.$$

If, moreover, Ψ is strongly u.s.c., then, in (b), $\mathrm{st}[\Phi, \mathscr{V}]$ can be replaced by $\mathrm{st}[\Psi, \mathscr{V}]$.

Proof. For (b) $\Rightarrow$ (a), let $\mathscr{F}$ be a locally-finite family of closed subsets of Y. Then, Y has an open cover $\mathscr{V}$ such that each $V \in \mathscr{V}$ meets only finitely many members of $\mathscr{F}$. By (b), there exists $Z \subset \ell_1(\mathscr{V})$, a u.s.c. $\theta : Z \to \mathscr{F}(Y)$ and a continuous $g : X \to Z$ such that $\mathrm{tb}(\theta) < \mathscr{V}$ and $\Psi(x) \subset \theta(g(x))$, $x \in X$. Take a point $z \in Z$. Then, $\theta(z)$ is covered by finitely many members of $\mathscr{V}$ because $\mathrm{tb}(\theta) < \mathscr{V}$ and, according to the properties of $\mathscr{V}$, only finitely many members of $\mathscr{F}$ may intersect $\theta(z)$. Since θ is u.s.c. and nonempty-valued, this implies that

$$U = \theta^{\#}\Bigl[Y \setminus \bigcup\bigl\{F \in \mathscr{F}\colon\ F \cap \theta(z) = \varnothing\bigr\}\Bigr]$$

is a neighbourhood of z which meets only finitely many $\theta^{-1}[F]$, $F \in \mathscr{F}$, so $\{\theta^{-1}[F]\colon F \in \mathscr{F}\}$ is a locally-finite family of closed subsets of Z. Since Z is countably paracompact and collectionwise normal (being metrizable), by a result of Dowker [3] (see, also, [4, 5.5.17]), there exists a locally-finite family $\{V_F\colon F \in \mathscr{F}\}$ of open subsets of Z such that $\theta^{-1}[F] \subset V_F$ for all $F \in \mathscr{F}$. However, $g : X \to Z$ is continuous and $\Psi(x) \subset \theta(g(x))$ for all $x \in X$. Hence, $\{g^{-1}(V_F)\colon F \in \mathscr{F}\}$ is a locally-finite family of open subsets of X such that $\Psi^{-1}[F] \subset g^{-1}(\theta^{-1}[F]) \subset g^{-1}(V_F)$, $F \in \mathscr{F}$.

(a) ⇒ (b). Suppose Ψ has the locally-finite lifting property and $\mathscr{V}$ is an open cover of Y. Since Y is paracompact, it has a locally-finite cover $\{F_V\colon V \in \mathscr{V}\}$ consisting of zero-sets such that $F_V \subset V$ for all $V \in \mathscr{V}$. Let $H_V = \Psi^{-1}[F_V]$, $V \in \mathscr{V}$. Since Ψ is a nonempty-valued u.s.c. multi-selection of Φ, $\{H_V\colon V \in \mathscr{V}\}$ is a closed cover of X which refines $\{\Phi^{-1}[V]\colon V \in \mathscr{V}\}$, while $\{\Phi^{-1}[V]\colon V \in \mathscr{V}\}$ is an open cover of X because Φ is l.s.c. Since Ψ has the locally-finite lifting property, there is a locally-finite open cover $\{U_V\colon V \in \mathscr{V}\}$ of X with $H_V \subset U_V \subset \Phi^{-1}[V]$, $V \in \mathscr{V}$. For every $V \in \mathscr{V}$, take a continuous function $\xi_V : X \to [0,1]$ such that

$$H_V \subset G_V = \xi_V^{-1}(1) \subset \xi_V^{-1}\big((0,1]\big) \subset U_V. \tag{2.1}$$

Next, consider the continuous map $g : X \to \ell_1(\mathscr{V})$ defined by $g(x)(V) = \xi_V(x)$, $V \in \mathscr{V}$ and $x \in X$. By (2.1), g is well defined because $\{U_V\colon V \in \mathscr{V}\}$ is a locally-finite cover of X. For convenience, let $Z = g(X)$, and let $\rho(z_1,z_2) = \|z_1 - z_2\|$, $z_1, z_2 \in Z$, be the metric on Z generated by the norm of $\ell_1(\mathscr{V})$. In order to construct our $\theta : Z \to \mathscr{F}(Y)$, whenever $V \in \mathscr{V}$, let T_V be the closure of $g(H_V)$ in Z, and let us show that

$$g^{-1}(T_V) \subset G_V. \tag{2.2}$$

Indeed, take a point $x \in g^{-1}(T_V)$. Then, for every positive number $r > 0$ there exists $x_r \in H_V$ such that $\rho(g(x), g(x_r)) < r$, so

$$\big|\xi_V(x) - 1\big| = \big|\xi_V(x) - \xi_V(x_r)\big| \leqslant \rho\big(g(x), g(x_r)\big) < r.$$

Hence, $\xi_V(x) = 1$ which, by (2.1), implies that $x \in G_V$.

Now, let us show that $\{T_V\colon V \in \mathscr{V}\}$ is locally-finite in Z. Take a point $x \in X$, and let $\mathscr{V}(x) = \{V \in \mathscr{V}\colon \xi_V(x) > 1/2\}$ which is clearly a finite set. If $z \in B^{\rho}_{1/2}(g(x)) \cap T_V$ for some $V \in \mathscr{V}$, then $\rho(g(x), z) < 1/2$ and, by (2.2), there exists $t \in G_V$ such that $\rho(g(x), g(t)) < 1/2$. Thus, $\xi_V(x) > 1/2$ because, by (2.1),

$$\big|\xi_V(x) - 1\big| = \big|\xi_V(x) - \xi_V(t)\big| \leqslant \rho\big(g(x), g(t)\big) < 1/2.$$

That is, $V \in \mathscr{V}(x)$, and hence $B^{\rho}_{1/2}(g(x))$ intersects only finitely many elements of $\{T_V\colon V \in \mathscr{V}\}$. Finally, define the required $\theta : Z \to \mathscr{F}(Y)$ by

$$\theta(z) = \bigcup\{F_V\colon V \in \mathscr{V} \text{ and } z \in T_V\}, \quad z \in Z.$$

Since $\{T_V\colon V \in \mathscr{V}\}$ is a locally-finite closed cover of Z and each F_V, $V \in \mathscr{V}$, is closed, the mapping θ is u.s.c. and closed-valued. By the same reason, $\mathrm{tb}(\theta) < \mathscr{V}$. If $x \in X$ and $\mathscr{P}(x) = \{V \in \mathscr{V}\colon x \in H_V\}$, then, by (2.1) and (2.2),

$$\begin{aligned}\mathscr{P}(x) &\subset \big\{V \in \mathscr{V}\colon g(x) \in T_V\big\} \subset \{V \in \mathscr{V}\colon x \in U_V\} \\ &\subset \big\{V \in \mathscr{V}\colon x \in \Phi^{-1}[V]\big\} = \big\{V \in \mathscr{V}\colon V \cap \Phi(x) \neq \varnothing\big\}.\end{aligned}$$

Thus, $\Psi(x) \subset \theta(g(x)) \subset \mathrm{st}[\Phi, \mathscr{V}](x)$.

Finally, suppose that Ψ is strongly u.s.c. In this case, each H_V, $V \in \mathscr{V}$, will be a zero-set of X because F_V, $V \in \mathscr{V}$, are zero-sets of Y. Then, in (2.1), we may assume that $H_V = G_V$ for every $V \in \mathscr{V}$. Hence, by (2.2), we now get that

$$T_V = g(H_V) \text{ and } H_V = g^{-1}(T_V), \quad \text{for every } V \in \mathscr{V}. \tag{2.3}$$

Take a point $x \in X$. By (2.3), this implies that

$$\begin{aligned}\big\{V \in \mathscr{V}\colon g(x) \in T_V\big\} &= \{V \in \mathscr{V}\colon x \in H_V\} \\ &= \big\{V \in \mathscr{V}\colon \Psi(x) \cap F_V \neq \varnothing\big\} \\ &\subset \big\{V \in \mathscr{V}\colon \Psi(x) \cap V \neq \varnothing\big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\theta\big(g(x)\big) &= \bigcup\big\{F_V\colon V \in \mathscr{V} \text{ and } g(x) \in T_V\big\} \\ &\subset \bigcup\big\{V \in \mathscr{V}\colon \Psi(x) \cap V \neq \varnothing\big\} = \mathrm{st}[\Psi, \mathscr{V}](x). \qquad \square\end{aligned}$$

We proceed with several examples of usco mappings having the locally-finite lifting property. Recall that, for an infinite cardinal number τ, a space X is called *τ-collectionwise normal* if every discrete family $\mathscr{F}$ of closed subsets of X, with

$|\mathscr{F}| \leqslant \tau$, has a discrete family $\{U_F\colon F \in \mathscr{F}\}$ of open subsets of X such that $F \subset U_F$ for each $F \in \mathscr{F}$. A space X is collectionwise normal if it is τ-collectionwise normal for every τ. It is well known that X is normal if and only if it is ω-collectionwise normal. However, for every τ there exists a τ-collectionwise normal space which is not τ^+-collectionwise normal [23], where the cardinal τ^+ is the immediate successor of τ. According to a result of Dowker [3] (see, also, [4, 5.5.17]), a normal space X is countably paracompact and τ-collectionwise normal if and only if X is τ-expandable [13]; that is, every locally-finite family $\mathscr{F}$ of closed subsets of X, with $|\mathscr{F}| \leqslant \tau$, has a locally-finite family $\{U_F\colon F \in \mathscr{F}\}$ of open subsets of X such that $F \subset U_F$ for each $F \in \mathscr{F}$.

Proposition 2.2. *For a normal space X and an infinite cardinal number τ, the following are equivalent*:

(a) *X is τ-expandable.*
(b) *If Y is a metrizable space with $w(Y) \leqslant \tau$, then every usco mapping $\Psi : X \to \mathscr{C}(Y)$ has the locally-finite lifting property.*

Proof. The implication (a) $\Rightarrow$ (b) follows by [8, Example 3.9]. As for (b) $\Rightarrow$ (a), let $\mathscr{F}$ be a locally-finite family of closed subsets of X, with $|\mathscr{F}| \leqslant \tau$. Endowing $Y = \{\mathscr{F}\} \cup \mathscr{F}$ with the discrete topology, define an usco mapping $\Psi : X \to \mathscr{C}(Y)$ by $\Psi(x) = \{\mathscr{F}\} \cup \{F \in \mathscr{F}\colon x \in F\}$, $x \in X$. Since $\{\{F\}\colon F \in \mathscr{F}\}$ is a locally-finite family of closed subsets of Y, by (b), there exists a locally-finite family $\{U_F\colon F \in \mathscr{F}\}$ of open subsets of X such that $F = \Psi^{-1}[\{F\}] \subset U_F$, $F \in \mathscr{F}$. Hence, X is τ-expandable. $\square$

In what follows, for a subset $M \subset X$ and a collection $\mathscr{W}$ of subsets of X, we will use $\mathrm{Ord}(\mathscr{W}, M)$ to denote the *order* of $\mathscr{W}$ in M, i.e. the smallest cardinal number $\varkappa$ such that $|\{W \in \mathscr{W}\colon x \in W\}| \leqslant \varkappa$ for every $x \in M$. For a natural number $m \in \mathbb{N}$, let $\mathscr{C}_m(Y) = \{S \in \mathscr{C}(Y)\colon |S| \leqslant m\}$. By a *finite-dimensional* space, we mean a space which has a finite covering dimension.

Proposition 2.3. *For a normal space X and an infinite cardinal number τ, the following are equivalent*:

(a) *X is τ-collectionwise normal.*
(b) *If Y is a finite-dimensional metrizable space, with $w(Y) \leqslant \tau$, and $m \in \mathbb{N}$, then every usco mapping $\Psi : X \to \mathscr{C}_m(Y)$ has the locally-finite lifting property.*

Proof. The implication (a) $\Rightarrow$ (b) follows by [8, Example 3.10]. To show that (b) $\Rightarrow$ (a), let $\mathscr{F}$ be a locally-finite family of closed subsets of X such that $|\mathscr{F}| \leqslant \tau$ and $\mathrm{Ord}(\mathscr{F}, X) \leqslant n$ for some $n \in \mathbb{N}$. As before, endow $Y = \{\mathscr{F}\} \cup \mathscr{F}$ with the discrete topology, and define an usco mapping $\Psi : X \to \mathscr{C}(Y)$ by letting for $x \in X$ that $\Psi(x) = \{\mathscr{F}\} \cup \{F \in \mathscr{F}\colon x \in F\}$. Then, $\Psi : X \to \mathscr{C}_{n+1}(Y)$ and $\{\{F\}\colon F \in \mathscr{F}\}$ is a locally-finite family of closed subsets of Y. Hence, by (b), there is a locally-finite family $\{U_F\colon F \in \mathscr{F}\}$ of open subsets of X with $F = \Psi^{-1}[\{F\}] \subset U_F$, $F \in \mathscr{F}$. By a result of Katětov [12], X is τ-collectionwise normal. $\square$

According to Proposition 2.2, the family "$\mathscr{C}_m(Y)$" in Proposition 2.3 cannot be replaced by "$\mathscr{C}(Y)$". However, the requirement that "Y is finite-dimensional" can be omitted if Ψ is strongly u.s.c.

Lemma 2.4. *Let X be a τ-collectionwise normal space, Y be a metrizable space, with $w(Y) \leqslant \tau$, and $m \in \mathbb{N}$. Then every strongly u.s.c. $\Psi : X \to \mathscr{C}_m(Y)$ has the locally-finite lifting property.*

Proof. Let $\mathscr{F}$ be a locally-finite family consisting of nonempty closed subsets of Y. We are going to construct a locally-finite open cover $\{W_k\colon k < \omega\}$ of X and a closed cover $\{M_k\colon k < \omega\}$ of X such that, for every $k < \omega$,

$$M_k \subset W_k, \tag{2.4}$$

$$\mathrm{Ord}\big(\big\{\Psi^{-1}[F]\colon F \in \mathscr{F}\big\}, M_k\big) < \omega. \tag{2.5}$$

Note that this will be sufficient. Indeed, because of τ-collectionwise normality of X, by (2.4), (2.5) and [12], there will be a locally-finite family $\{U_{(k,F)}\colon F \in \mathscr{F}\}$ of open subsets of X such that $\Psi^{-1}[F] \cap M_k \subset U_{(k,F)} \subset W_k$ for all $F \in \mathscr{F}$. Set $U_F = \bigcup\{U_{(k,F)}\colon k < \omega\}$. Then, $\{U_F\colon F \in \mathscr{F}\}$ is a locally-finite family of open subsets of X such that $\Psi^{-1}[F] \subset U_F$, $F \in \mathscr{F}$.

So, it only remains to construct these W_k's and M_k's. To this end, for every $y \in Y$, let $f(y) = |\{F \in \mathscr{F}\colon y \in F\}|$. Then, $\{V_k\colon k < \omega\}$ is an increasing open cover of Y, where $V_k = \{y \in Y\colon f(y) \leqslant k\}$, $k < \omega$. Since Ψ is compact-valued, $X = \bigcup\{\Psi^\#[V_k]\colon k < \omega\}$. Since Ψ is strongly u.s.c., each $\Psi^\#[V_k]$ is a cozero-set of X. Hence, there exists a locally-finite open cover $\{W_k\colon k < \omega\}$ of X with $W_k \subset \Psi^\#[V_k]$, $k < \omega$. Finally, take a closed cover $\{M_k\colon k < \omega\}$ of X such that $M_k \subset W_k$, $k < \omega$. To show that (2.5) holds, let us check that $\mathrm{Ord}(\{\Psi^{-1}[F]\colon F \in \mathscr{F}\}, M_k) \leqslant mk$. Indeed, take $x \in M_k$. Then,

$$\big|\big\{F \in \mathscr{F}\colon x \in \Psi^{-1}[F]\big\}\big| = \big|\big\{F \in \mathscr{F}\colon \Psi(x) \cap F \neq \varnothing\big\}\big| = \sum\big\{f(y)\colon y \in \Psi(x)\big\} \leqslant mk$$

because $|\Psi(x)| \leqslant m$ and $\Psi(x) \subset V_k$. The proof is completed. $\square$

For an infinite cardinal number τ, a space X is called *τ-PF-normal* (see [24]) if every point-finite open cover of cardinality $\leqslant \tau$ is normal. Every collectionwise normal space is τ-PF-normal for every cardinal τ [14], and ω-PF-normality coincides with normality [19]. However, PF-normality is neither identical with collectionwise normality (see Bing's example in [1] and [14, Example 1]), nor with normality (see [14, Example 1]). For some properties of PF-normal spaces, the interested reader is referred to [9, Section 3] and [14].

Proposition 2.5. *For a normal space X and an infinite cardinal number τ, the following are equivalent*:

(a) *X is τ-PF-normal.*
(b) *If Y is a metrizable space, with $w(Y) \leqslant \tau$, and $\Phi : X \to \mathscr{C}(Y)$ is l.s.c., then every usco multi-selection $\Psi : X \to \mathscr{C}(Y)$ of Φ has the locally-finite lifting property.*

Proof. To show that (a) $\Rightarrow$ (b), suppose that X is τ-PF-normal, Y is a metrizable space, with $w(Y) \leqslant \tau$, $\Phi : X \to \mathscr{C}(Y)$ is l.s.c., $\Psi : X \to \mathscr{C}(Y)$ is an usco multi-selection for Φ, and $\mathscr{F}$ is a locally-finite family of closed subsets of Y. Since $w(Y) \leqslant \tau$, we have $|\mathscr{F}| \leqslant \tau$. By a result of Dowker [3], there is a locally-finite family $\{V_F\colon F \in \mathscr{F}\}$ of open subsets of Y such that $F \subset V_F$ for each $F \in \mathscr{F}$. Since Φ is l.s.c. and compact-valued, $\{\Phi^{-1}[V_F]\colon F \in \mathscr{F}\}$ is a point-finite family of open subsets of X, while $\{\Psi^{-1}[F]\colon F \in \mathscr{F}\}$ is a locally-finite family of closed subsets of X with $\Psi^{-1}[F] \subset \Phi^{-1}[V_F]$, $F \in \mathscr{F}$, because Ψ is an usco multi-selection of Φ. Then, by [9, Theorem 3.1], X has a locally-finite open family $\{U_F\colon F \in \mathscr{F}\}$ such that $\Psi^{-1}[F] \subset U_F$, $F \in \mathscr{F}$, so Ψ has the locally-finite lifting property.

To show that (b) $\Rightarrow$ (a), let X be as in (b). We follow the previous proofs. Briefly, let $\mathscr{F}$ be a locally-finite family of closed subsets of X, with $|\mathscr{F}| \leqslant \tau$, and let $\{U_F\colon F \in \mathscr{F}\}$ be a point-finite family of open subsets of X such that $F \subset U_F$, $F \in \mathscr{F}$. Endowing $Y = \{\mathscr{F}\} \cup \mathscr{F}$ with the discrete topology, define an l.s.c. mapping $\Phi : X \to \mathscr{C}(Y)$ by $\Phi(x) = \{\mathscr{F}\} \cup \{F \in \mathscr{F}\colon x \in U_F\}$, $x \in X$, and an usco mapping $\Psi : X \to \mathscr{C}(Y)$ by $\Psi(x) = \{\mathscr{F}\} \cup \{F \in \mathscr{F}\colon x \in F\}$, $x \in X$. Since Ψ is a multi-selection of Φ and $\{\{F\}\colon F \in \mathscr{F}\}$ is a locally-finite family of closed subsets of Y, by (b), there is a locally-finite family $\{U_F\colon F \in \mathscr{F}\}$ of open subsets of X with $F = \Psi^{-1}[\{F\}] \subset U_F$, $F \in \mathscr{F}$. By [9, Theorem 3.1], X is τ-PF-normal. $\square$

3. Intermediate usco factorisations

Motivated by weak-factorisations of l.s.c. mappings (see the Introduction) and Theorem 2.1, we are going to consider special weak-factorisations of u.s.c. mappings. Namely, let $(\Psi, \Phi) : X \to \mathscr{F}(Y)$ be a pair of mappings such that Φ is l.s.c., and Ψ is a u.s.c. multi-selection for Φ. We shall say that a triple (Z, g, ψ) is an *intermediate u.s.c. weak-factorisation* of (Ψ, Φ) provided that Z is a metrizable space with $w(Z) \leqslant w(Y)$, $g : X \to Z$ is continuous, and $\psi : Z \to \mathscr{F}(Y)$ is u.s.c. such that $\Psi(x) \subset \psi(g(x)) \subset \Phi(x)$ for every $x \in X$. In particular, we will say that (Z, g, ψ) is an *intermediate usco weak-factorisation* if ψ is an usco mapping (in which case, of course, Ψ must be also usco).

Theorem 3.1. *Let X be a normal space, Y be a completely metrizable space, $\Phi : X \to \mathscr{F}(Y)$ be l.s.c., and $\Psi : X \to \mathscr{C}(Y)$ be an usco multi-selection for Φ which has the locally-finite lifting property. Then, (Ψ, Φ) has an intermediate usco weak-factorisation.*

Turning to the proof of Theorem 3.1, for $\varepsilon > 0$ and a subset $A \subset Y$ of a metric space (Y, d), we write $\mathsf{tb}_d(A) < \varepsilon$ if there exists a finite subset $F \subset A$ with $A \subset B^d_\varepsilon(F)$, see [11]. Here, "$\mathsf{tb}_d(A)$" plays the role of the infimum of all such $\varepsilon > 0$. In particular, we have that $\mathsf{tb}_d(A) = 0$ if $\mathsf{tb}_d(A) < \varepsilon$ for every $\varepsilon > 0$. In this case, A will be totally bounded with respect to d. Hence, if (Y, d) is complete, then a closed subset $A \subset Y$ is compact if and only if $\mathsf{tb}_d(A) = 0$. Also, note that if $\mathscr{V}$ is a cover of Y with $\mathsf{diam}_d(V) < \varepsilon$, $V \in \mathscr{V}$, then $\mathsf{tb}(A) < \mathscr{V}$ implies $\mathsf{tb}_d(A) < \varepsilon$. Finally, for a mapping $\Phi : X \to 2^Y$, we write that $\mathsf{tb}_d(\Phi) < \varepsilon$ if $\mathsf{tb}_d(\Phi(x)) < \varepsilon$ for every $x \in X$.

The key element in the proof of Theorem 3.1 is the following proposition dealing with a countable intersection of closed-valued u.s.c. mappings. It should be mentioned that, in general, such an intersection is not necessarily u.s.c.

Proposition 3.2. *Let Z be a space, (Y, d) be a complete metric space, and let $\psi_n : Z \to \mathscr{F}(Y)$, $n < \omega$, be u.s.c. mappings such that*

(a) $\lim_{n\to\infty} \mathsf{tb}_d(\psi_n) = 0$,
(b) *each family $\{\psi_n(z)\colon n < \omega\}$, $z \in Z$, has the finite intersection property.*

Define $\psi : Z \to 2^Y$ by $\psi(z) = \bigcap\{\psi_n(z)\colon n < \omega\}$, $z \in Z$. Then, ψ is an usco mapping.

Proof. For every $n < \omega$, define a mapping $\theta_n : Z \to \mathscr{F}(Y)$ by

$$\theta_n(z) = \bigcap\big\{\psi_k(z)\colon k \leqslant n\big\}, \quad z \in Z.$$

By (b), each θ_n, $n < \omega$, is well defined and is u.s.c. as a finite intersection of u.s.c. mappings (see [4, 1.7.17]). Also, $\mathsf{tb}_d(\theta_n) \leqslant \mathsf{tb}_d(\psi_n)$, $n < \omega$. Hence, by (a), $\lim_{n\to\infty} \mathsf{tb}_d(\theta_n) = 0$. Since $\psi(z) = \bigcap\{\theta_n(z)\colon n < \omega\}$, $z \in Z$, by (a), (b) and [6, Lemma 3.2], ψ is nonempty-compact-valued. To show that it is u.s.c., take a point $z \in Z$ and $\varepsilon > 0$. By [6, Lemma 3.2] once again, there exists an $m < \omega$ such that $\theta_m(z) \subset B^d_{\varepsilon/2}(\psi(z))$. Since θ_m is u.s.c., $V = \theta_m^{\#}[B^d_{\varepsilon/2}(\theta_m(z))]$ is an open set containing z. If $x \in V$, then

$$\psi(x) \subset \bigcap\{\theta_n(x)\colon n < \omega\} \subset \theta_m(x) \subset B^d_{\varepsilon/2}\big(\theta_m(z)\big) \subset B^d_{\varepsilon}\big(\psi(z)\big). \qquad \square$$

Proof of Theorem 3.1. Let $\{\mathscr{V}_n\colon n < \omega\}$ be a sequence of locally-finite open covers of Y such that $\operatorname{diam}_d(V) < 2^{-n}$, $V \in \mathscr{V}_n$ and $n < \omega$, with respect to a complete metric d on Y. For each $n < \omega$, by Theorem 2.1, there exists a subset $Z_n \subset \ell_1(\mathscr{V}_n)$, a continuous surjective $g_n : X \to Z_n$, and a u.s.c. $\theta_n : Z_n \to \mathscr{F}(Y)$ such that $\operatorname{tb}_d(\theta_n) < 2^{-n}$ and $\Psi(x) \subset \theta_n(g_n(x)) \subset B^d_{2^{-n}}(\Phi(x))$, $x \in X$. In particular, we have $w(Z_n) \leqslant w(\ell_1(\mathscr{V}_n)) \leqslant |\mathscr{V}_n| \leqslant w(Y)$, $n < \omega$, because each family $\mathscr{V}_n$, $n < \omega$, is locally-finite. Let $Z = \prod\{Z_n\colon n < \omega\}$ and, for every $n < \omega$, let $\pi_n : Z \to Z_n$ be the projection. Also, let $g = \Delta\{g_n\colon n < \omega\} : X \to Z$ be the diagonal map. Finally, for every $n < \omega$, define a mapping $\psi_n : Z \to \mathscr{F}(Y)$ by $\psi_n(z) = \theta_n(\pi_n(z))$, $z \in Z$. Then, $\psi_n : Z \to \mathscr{F}(Y)$ is u.s.c. and $\Psi(x) \subset \theta_n(\pi_n(g(x))) = \psi_n(g(x))$ for every $x \in X$. Hence, each family $\{\psi_n(z)\colon n < \omega\}$, $z \in Z$, has the finite intersection property. Since $\lim_{n\to\infty} \operatorname{tb}_d(\psi_n) = \lim_{n\to\infty} \operatorname{tb}_d(\theta_n) = 0$, by Proposition 3.2, $\psi(z) = \bigcap\{\psi_n(z)\colon n < \omega\}$, $z \in Z$, defines an usco mapping. If $x \in X$, then

$$\begin{aligned}\Psi(x) \subset \psi\big(g(x)\big) = \bigcap\big\{\psi_n\big(g(x)\big)\colon n < \omega\big\} = \bigcap\big\{\theta_n\big(g_n(x)\big)\colon n < \omega\big\} \\ \subset \bigcap\big\{B^d_{2^{-n}}\big(\Phi(x)\big)\colon n < \omega\big\} \subset \Phi(x). \qquad \square\end{aligned}$$

For convenience, for a u.s.c. mapping $\Psi : X \to \mathscr{F}(Y)$, we shall say that a triple (Z, g, ψ) is a *u.s.c. weak-factorisation* of Ψ if Z is a metrizable space with $w(Z) \leqslant w(Y)$, $g : X \to Z$ is continuous, and $\psi : Z \to \mathscr{F}(Y)$ is u.s.c. such that $\Psi(x) \subset \psi(g(x))$ for every $x \in X$. In this case, (Z, g, ψ) is a u.s.c. weak-factorisation of Ψ if and only if it is an intermediate u.s.c. weak-factorisation for the pair (Ψ, Φ), where $\Phi(x) = Y$, $x \in X$. Just like before, we shall say that (Z, g, ψ) is an *usco weak-factorisation* of Ψ if ψ is usco.

Now, we list several consequences of Theorem 3.1. First, by Theorem 3.1 and Proposition 2.2, we get the following immediate result.

Corollary 3.3. *Let X be a normal and τ-expandable space, Y be a completely metrizable space, with $w(Y) \leqslant \tau$, $\Phi : X \to \mathscr{F}(Y)$ be l.s.c., and let $\Psi : X \to \mathscr{C}(Y)$ be an usco multi-selection for Φ. Then, the pair (Ψ, Φ) has an intermediate usco weak-factorisation. In particular, every usco mapping $\Psi : X \to \mathscr{C}(Y)$ has an usco weak-factorisation.*

In the same way, by Theorem 3.1 and Proposition 2.3, we have the following consequence.

Corollary 3.4. *Let X be a τ-collectionwise normal space, Y be a completely metrizable finite-dimensional space, with $w(Y) \leqslant \tau$, $\Phi : X \to \mathscr{F}(Y)$ be l.s.c., $m \in \mathbb{N}$, and let $\Psi : X \to \mathscr{C}_m(Y)$ be an usco multi-selection for Φ. Then, the pair (Ψ, Φ) has an intermediate usco weak-factorisation. In particular, every usco mapping $\Psi : X \to \mathscr{C}_m(Y)$ has an usco weak-factorisation.*

Since every metrizable space can be embedded into a completely metrizable one, by Theorem 3.1 and Proposition 2.5, we have also the following consequence.

Corollary 3.5. *Let X be a τ-PF-normal space, Y be a metrizable space, with $w(Y) \leqslant \tau$, $\Phi : X \to \mathscr{C}(Y)$ be l.s.c., and let $\Psi : X \to \mathscr{C}(Y)$ be an usco multi-selection for Φ. Then, the pair (Ψ, Φ) has an intermediate usco weak-factorisation.*

4. Factorising strongly usco mappings

Recall from the Introduction that if Y is a metrizable space and $\Psi : X \to \mathscr{C}(Y)$ is usco, then a triple (Z, g, ψ) is an *usco factorisation* of Ψ if Z is a metrizable space with $w(Z) \leqslant w(Y)$, $g : X \to Z$ is continuous, and $\psi : Z \to \mathscr{C}(Y)$ is an usco mapping such that $\Psi(x) = \psi(g(x))$ for all $x \in X$.

Theorem 4.1. *Let X be a normal space, Y be a metrizable space, and let $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping having the locally-finite lifting property. Then, Ψ is strongly u.s.c. if and only if it has an usco factorisation.*

Proof. If Ψ has an usco factorisation (Z, g, ψ), then $\Psi^{-1}[F] = g^{-1}(\psi^{-1}[F])$ for every $F \subset Y$. In particular, for a zero-set $F \subset Y$, the set $\psi^{-1}[F]$ is a zero-set of Z because ψ is usco and Z is metrizable, hence $\Psi^{-1}[F]$ is a zero-set of X because g is continuous. Thus, Ψ is strongly u.s.c.

Conversely, let Ψ be strongly u.s.c. Take a metric d compatible with the topology of Y, and let $(\tilde{Y}, d)$ be the completion of (Y, d). Then, $\Psi : X \to \mathscr{C}(\tilde{Y})$ remains strongly u.s.c. and, by Theorem 2.1, Ψ has a sequence of u.s.c. weak-factorisations (Z_n, g_n, θ_n), $n < \omega$, such that each $g_n : X \to Z_n$ is surjective and $\theta_n(g_n(x)) \subset B^d_{2^{-n}}(\Psi(x))$ for every $x \in X$. Let $Z = \prod\{Z_n\colon n < \omega\}$, $\pi_n : Z \to Z_n$, $n < \omega$, be the projections, and $g = \Delta\{g_n\colon n < \omega\} : X \to Z$ be the diagonal map. Observe that

$w(Z) \leqslant w(Y)$ because $w(Z_n) \leqslant w(Y)$, $n < \omega$. Next, for every $n < \omega$ define a mapping $\psi_n : Z \to \mathscr{F}(\tilde{Y})$ by $\psi_n(z) = \theta_n(\pi_n(z))$, $z \in Z$. Then, each $\psi_n : Z \to \mathscr{F}(\tilde{Y})$, $n < \omega$, is u.s.c. and $\Psi(x) \subset \theta_n(\pi_n(g(x))) = \psi_n(g(x))$ for every $x \in X$. Hence, each family $\{\psi_n(z)\colon n < \omega\}$, $z \in Z$, has the finite intersection property. Since $\lim_{n\to\infty} \mathrm{tb}_d(\psi_n) = \lim_{n\to\infty} \mathrm{tb}_d(\theta_n) = 0$, by Proposition 3.2, we may define an usco mapping $\psi : Z \to \mathscr{C}(\tilde{Y})$ by $\psi(z) = \bigcap\{\psi_n(z)\colon n < \omega\}$, $z \in Z$. If $x \in X$, then

$$\begin{aligned}\Psi(x) \subset \psi\big(g(x)\big) &= \bigcap\big\{\psi_n\big(g(x)\big)\colon n < \omega\big\}\\ &= \bigcap\big\{\theta_n\big(g_n(x)\big)\colon n < \omega\big\}\\ &\subset \bigcap\big\{B^d_{2^{-n}}\big(\Psi(x)\big)\colon n < \omega\big\} \subset \Psi(x).\end{aligned}$$

That is, $\Psi(x) = \psi(g(x))$ for every $x \in X$, and, in particular, $\psi : Z \to \mathscr{C}(Y)$. Thus, (Z, g, ψ) is an usco factorisation of Ψ. □

Now, we turn to some consequences of Theorem 4.1. By Propositions 2.2 and 2.5 and Lemma 2.4, we get the following:

Corollary 4.2. *Let X be a normal space, Y be a metrizable space, and let $\Psi : X \to \mathscr{C}(Y)$ be a strongly u.s.c. mapping. Then, in each of the following cases, Ψ has an usco factorisation*:

(a) *X is τ-expandable and $w(Y) \leqslant \tau$;*
(b) *X is τ-PF-normal, $w(Y) \leqslant \tau$, and Ψ is a multi-selection for some l.s.c. $\Phi : X \to \mathscr{C}(Y)$;*
(c) *X is τ-collectionwise normal, $w(Y) \leqslant \tau$, and $\Psi : X \to \mathscr{C}_m(Y)$ for some $m \in \mathbb{N}$.*

Recall that a space X is *perfect* if every closed subset of X is a G_δ-set. By Theorem 4.1, we get also the following consequence.

Corollary 4.3. *A space X is perfect and τ-collectionwise normal if and only if for every metrizable space Y, with $w(Y) \leqslant \tau$, every usco mapping $\Psi : X \to \mathscr{C}(Y)$ has an usco factorisation.*

Proof. If X is perfect and τ-collectionwise normal, then it is countably paracompact, while every u.s.c. mapping is strongly u.s.c. Hence, this implication follows by Corollary 4.2. To show the converse, let X be a space with the condition in the statement. Take a closed subset $F \subset X$, and define an usco mapping $\Psi : X \to \mathscr{C}(\{0,1\})$ by $\Psi(x) = \{0,1\}$ if $x \in F$, and $\Psi(x) = \{0\}$ otherwise. By assumption, Ψ has an usco factorisation (Z, g, ψ). Since $\psi^{-1}[\{1\}]$ is closed in the metrizable space Z, it is a zero-set of Z, and hence $F = \Psi^{-1}[\{1\}] = g^{-1}(\psi^{-1}[\{1\}])$ is a zero-set of X. Thus, X is perfectly normal (see [4, Theorem 1.5.9]), and, in particular, normal. Finally, by Theorem 2.1 and Proposition 2.3, X is τ-collectionwise normal as well. □

5. Factorising usco mappings with a zero-graph

In this section, we relate usco factorisations to the graph of usco mappings.

Theorem 5.1. *Let X be a normal space, Y be a completely metrizable space, and let $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping having the locally-finite lifting property. Then, the following are equivalent*:

(a) *Ψ has a zero-graph.*
(b) *There is a sequence $\Phi_n : X \to \mathscr{F}(Y)$, $n < \omega$, of l.s.c. mappings such that $\Psi(x) = \bigcap\{\Phi_n(x)\colon n < \omega\}$ for every $x \in X$.*
(c) *Ψ has an usco factorisation.*

Proof. Suppose that $\mathrm{Graph}(\Psi)$ is a zero-set in $X \times Y$. Then, there exists a decreasing sequence $\{O_n\colon n < \omega\}$ of open sets of $X \times Y$ such that $\mathrm{Graph}(\Psi) = \bigcap_{n<\omega} \overline{O_n}$. In particular, each O_n, $n < \omega$, defines an open-graph mapping $\Omega_n : X \to 2^Y$ for which $\mathrm{Graph}(\Omega_n) = O_n$. So, for every $n < \omega$, we may define an l.s.c. mapping $\Phi_n : X \to \mathscr{F}(Y)$ by $\Phi_n(x) = \overline{\Omega_n(x)}$, $x \in X$, see [15, Proposition 2.3]. Thus, we have that $\Psi(x) = \bigcap_{n<\omega} \Phi_n(x)$, $x \in X$, which is (a) ⇒ (b). To show that (b) ⇒ (c), suppose that $\Phi_n : X \to \mathscr{F}(Y)$, $n < \omega$, are as in (b). By Theorem 3.1, each pair (Ψ, Φ_n), $n < \omega$, has an intermediate usco weak-factorisation (Z_n, g_n, ψ_n) such that $g_n : X \to Z_n$ is surjective. Let $Z = \prod\{Z_n\colon n < \omega\}$, and let $g = \Delta\{g_n\colon n < \omega\} : X \to Z$ be the diagonal map. Also, for every $n < \omega$, let $\pi_n : Z \to Z_n$ be the projection. Finally, define $\psi : Z \to \mathscr{C}(Y)$ by

$$\psi(z) = \bigcap\big\{\psi_n\big(\pi_n(z)\big)\colon n < \omega\big\}, \quad z \in Z,$$

which is possible because each g_n, $n < \omega$, is surjective and $\Psi(x) \subset \psi_n(\pi_n(g(x)))$, $x \in X$. This ψ is usco as an intersection of the usco mappings $\psi_n \circ \pi_n : Z \to \mathscr{C}(Y)$, $n < \omega$, see, e.g., [4, 3.12.28]. If $x \in X$, then $\Psi(x) \subset \psi(g(x))$ and

$$\begin{aligned}\psi\big(g(x)\big) &= \bigcap\big\{\psi_n\big(\pi_n\big(g(x)\big)\big)\colon n<\omega\big\}\\ &= \bigcap\big\{\psi_n\big(g_n(x)\big)\colon n<\omega\big\} \subset \bigcap\big\{\Phi_n(x)\colon n<\omega\big\} = \Psi(x).\end{aligned}$$

That is, (c) holds. Since (c) ⇒ (a) follows by Proposition 1.1, the proof is completed. □

First of all, let us explicitly mention the following consequence of Theorem 5.1.

Corollary 5.2. *For a normal space X, a completely metrizable space Y, and an usco mapping $\Psi : X \to \mathscr{C}(Y)$ having the locally-finite lifting property, the following are equivalent*:

(a) *There is a sequence $\Phi_n : X \to \mathscr{C}(Y)$, $n<\omega$, of l.s.c. mappings such that $\Psi(x) = \bigcap\{\Phi_n(x)\colon n<\omega\}$ for every $x \in X$.*
(b) *Ψ has an usco factorisation (Z, g, ψ).*
(c) *Ψ has a zero-graph and is a multi-selection for some l.s.c. $\Phi : X \to \mathscr{C}(Y)$.*

Proof. The implication (a) ⇒ (b) follows by Theorem 5.1. For (b) ⇒ (c), suppose that Ψ has an usco factorisation (Z, g, ψ). By Proposition 1.1, Ψ has a zero-graph. Turning to the second part, take in mind that Z is countably paracompact and collectionwise normal (being metrizable). Hence, by [18, Theorem 1.3], ψ is a multi-selection of some l.s.c. mapping $\varphi : Z \to \mathscr{C}(Y)$. Then, the mapping $\Phi : X \to \mathscr{C}(Y)$ defined by $\Phi(x) = \varphi(g(x))$, $x \in X$, is as required. To show finally that (c) ⇒ (a), suppose that Ψ has a zero-graph and is a multi-selection for some l.s.c. $\Phi : X \to \mathscr{C}(Y)$. Then, $\mathsf{Graph}(\Psi) = \bigcap_{n<\omega} \overline{O_n}$ for some (decreasing) sequence of open sets $\{O_n \subset X \times Y\colon n<\omega\}$. In particular, each O_n, $n<\omega$, defines an open-graph mapping $\Omega_n : X \to 2^Y$ for which $\mathsf{Graph}(\Omega_n) = O_n$. So, for every $n<\omega$, we may define an l.s.c. mapping $\Phi_n : X \to \mathscr{C}(Y)$ by $\Phi_n(x) = \overline{\Phi(x) \cap \Omega_n(x)}$, $x \in X$, see [15, Propositions 2.3 and 2.4]. Thus, $\Psi(x) = \bigcap_{n<\omega} \Phi_n(x)$, $x \in X$. □

Just like before, we have a list of consequences from Theorem 5.1. Namely, by this theorem and Corollary 4.2, we get the following result.

Corollary 5.3. *Let X be a normal space, Y be a completely metrizable space, and let $\Psi : X \to \mathscr{C}(Y)$ be strongly u.s.c. Then, in each of the following cases, Ψ has a zero-graph*:

(a) *X is τ-expandable and $w(Y) \leqslant \tau$;*
(b) *X is τ-PF-normal, $w(Y) \leqslant \tau$ and Ψ is a multi-selection for some l.s.c. $\Phi : X \to \mathscr{C}(Y)$;*
(c) *X is τ-collectionwise normal, $w(Y) \leqslant \tau$ and $\Psi : X \to \mathscr{C}_m(Y)$ for some $m \in \mathbb{N}$.*

Concerning the possible relationship between strongly usco mappings and usco mappings with a zero-graph, we have the following questions.

Question 1. Let X be a (countably paracompact and collectionwise normal) space, Y be a metrizable space, and let $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping with a zero-graph. Is it true that Ψ is strongly u.s.c.?

Question 2. Let X be a (countably paracompact and collectionwise normal) space, (Y, d) be a metric space, and let $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping with a zero-set graph. If $(\tilde{Y}, d)$ is the completion of (Y, d), then is it true that $\Psi : X \to \mathscr{C}(\tilde{Y})$ has also a zero-graph in $X \times \tilde{Y}$?

It is well known that every usco mapping has a closed graph. Related to this, we have the following further question.

Question 3. Let X be a (countably paracompact and collectionwise normal) space, Y be a (completely) metrizable space, and $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping with a G_δ-graph. Then, is it true that Ψ has a zero-graph?

We have a partial answer to this question based on the following observation.

Proposition 5.4. *Let X be a paracompact space, Y be a regular space, $\Omega : X \to 2^Y$ be an open-graph mapping, and let $\Psi : X \to \mathscr{C}(Y)$ be an usco multi-selection for Ω. Then, there is an l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ such that $\Psi(x) \subset \Phi(x) \subset \Omega(x)$ for every $x \in X$.*

Proof. Take a point $x \in X$. Since Ω has an open graph, Ψ is u.s.c., $\Psi(x)$ is compact and Y is regular, there are open sets $V_x \subset X$ and $W_x \subset Y$ such that $x \in V_x \subset \Psi^{\#}[W_x]$ and $V_x \times \overline{W_x} \subset \mathsf{Graph}(\Omega)$. Since X is paracompact, $\{V_x\colon x \in X\}$ has a locally-finite open refinement $\mathscr{U}$. For every $U \in \mathscr{U}$ take a point $x(U) \in X$, with $U \subset V_{x(U)}$, and set $H_U = \overline{W_{x(U)}}$. Then,

$\Psi(z) \subset H_U \subset \Omega(z)$ for every $z \in U \in \mathscr{U}$. Since $\mathscr{U}$ is an open and locally-finite cover of X, we may define the required l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ by

$$\Phi(x) = \bigcup\{H_U\colon\ U \in \mathscr{U} \text{ and } x \in U\}, \quad x \in X. \qquad \square$$

According to Theorem 5.1, Proposition 5.4 implies the following immediate consequence.

Corollary 5.5. *Let X be a paracompact space, Y be a completely metrizable space, and $\Psi : X \to \mathscr{C}(Y)$ be an usco mapping. Then, Ψ has a zero-graph if and only if it has a G_δ-graph.*

Let us explicitly mention that Corollary 5.5 was obtained as an element in the proof of [25, Theorem 1.3] (see [25, Lemma 2.2]) using somewhat different technique based on the idea in the proof of Urysohn's lemma by directly constructing a continuous function $f : X \times Y \to [0, 1]$ with $\mathsf{Graph}(\Psi) = f^{-1}(1)$.

6. Continuous expansions and factorising usco mappings

A sequence $\Phi_n : X \to 2^Y$, $n < \omega$, of set-valued mapping is *decreasing* if each Φ_{n+1} is a multi-selection for Φ_n, $n < \omega$. According to the proof of (c) $\Rightarrow$ (a) of Corollary 5.2, we have the following separate result.

Proposition 6.1. *Let X and Y be spaces, and let $\Psi : X \to \mathscr{C}(Y)$ be a mapping with a zero-graph which is a multi-selection for some l.s.c. $\Phi : X \to \mathscr{C}(Y)$. Then, there is a decreasing sequence $\Phi_n : X \to \mathscr{C}(Y)$, $n < \omega$, of l.s.c. mappings such that $\Psi(x) = \bigcap\{\Phi_n(x)\colon\ n < \omega\}$ for every $x \in X$.*

Here, we relate a special case of Proposition 6.1 to usco factorisations. Turning to this, let us recall that a mapping $\psi : X \to \mathscr{C}(Y)$ is *upper δ-continuous* [10] if there is a sequence $\varphi_n : X \to \mathscr{C}(Y)$, $n < \omega$, of continuous (i.e., both l.s.c. and u.s.c.) mappings such that $\psi(x) = \bigcap\{\varphi_n(x)\colon\ n < \omega\}$, $x \in X$. For a topological vector space Y, a mapping $\psi : X \to \mathscr{C}(Y)$ is called *σ-selectionable* (see [22]) if there is a decreasing sequence $\varphi_n : X \to \mathscr{C}(Y)$, $n < \omega$, of continuous convex-valued mappings such that $\psi(x) = \bigcap\{\varphi_n(x)\colon\ n < \omega\}$, $x \in X$. Every σ-selectionable mapping is upper δ-continuous. In fact, for a Banach range, the converse is also true.

Proposition 6.2. *Let X be a space, Y be a Banach space, and $\Psi : X \to \mathscr{C}(Y)$ be a convex-valued upper δ-continuous mapping. Then, Ψ is σ-selectionable.*

Proof. By [10, Lemma 4.7], there exists a metrizable space Z, an upper δ-continuous $\psi : Z \to \mathscr{C}(Y)$ and a continuous $g : X \to Z$ such that $\psi(g(x)) = \Psi(x)$ for every $x \in X$. Hence, we may assume that X is itself metrizable. Then, Ψ is an usco mapping with a zero-graph and, by [25, Theorem 3.7], it must be σ-selectionable. $\square$

Since an arbitrary intersection of usco mappings is usco, every upper δ-continuous mapping $\Psi : X \to \mathscr{C}(Y)$ is an usco mapping which is a multi-selection for some continuous $\Phi : X \to \mathscr{C}(Y)$. According to a result of Nepomnyashchii [21] (see, also, [10, Corollary 2.10]), if X is a paracompact space, Y is a connected and locally connected completely metrizable space, and $\Psi : X \to \mathscr{C}(Y)$ is an usco mapping, then Ψ is a multi-selection for some continuous $\Phi : X \to \mathscr{C}(Y)$. On the other hand, there exists an usco mapping $\Psi : \mathbb{R} \to \mathscr{C}(\mathbb{N})$ which is not a multi-selection of any continuous $\Phi : \mathbb{R} \to \mathscr{C}(\mathbb{N})$, [10, Example 2.12], this is also true if $\mathbb{N}$ is replaced by the long topology sine curve [10, Example 2.13]. However, in each of these cases, if the range is embedded in a Banach space E, then $\Psi : X \to \mathscr{C}(E)$ will remain usco and will be a multi-selection for some continuous $\Phi : \mathbb{R} \to \mathscr{C}(E)$. Finally, let us also mention that if $\Psi : X \to \mathscr{C}(Y)$ is upper δ-continuous and Y is embedded in another space E, then $\Psi : X \to \mathscr{C}(E)$ remains upper δ-continuous. We now have the following result which is a generalisation of [10, Lemma 4.7].

Theorem 6.3. *Let X be a space, and Y be a metrizable space. Then, for a mapping $\Psi : X \to \mathscr{C}(Y)$, the following are equivalent*:

(a) *Whenever Y is embedded in a Banach space E, $\Psi : X \to \mathscr{C}(E)$ is upper δ-continuous.*
(b) *For some embedding of Y in a metrizable space E, $\Psi : X \to \mathscr{C}(E)$ is upper δ-continuous.*
(c) *Ψ has an usco factorisation.*

To prepare for the proof of Theorem 6.3, we proceed with some observations.

Proposition 6.4. *Let M be a paracompact space, E be a Banach space, $\mathscr{W}$ be a finite family of open convex subsets of E, and let $\{H_W\colon\ W \in \mathscr{W}\}$ be a family of closed subsets of E which refines $\mathscr{W}$. Also, let $\varphi : M \to \mathscr{F}(E)$ be an l.s.c. convex-valued mapping, and $\theta : M \to \mathscr{C}(E)$ be an usco multi-selection for φ such that, for every $x \in M$, $\theta(x) \subset \bigcup\{H_W\colon\ W \in \mathscr{W}\}$ and $\varphi(x) \cap H_W \neq \varnothing$, $W \in \mathscr{W}$. Then, φ has a continuous multi-selection $\psi : X \to \mathscr{C}(E)$ with $\theta(x) \subset \psi(x) \subset \overline{\bigcup \mathscr{W}}$ for every $x \in M$.*

Proof. For every $W \in \mathscr{W}$, let $A_W = \theta^{-1}[H_W]$ and $\varphi_W(x) = \overline{\varphi(x) \cap W}$, $x \in M$. Then, A_W is closed in M because θ is usco, while $\varphi_W : M \to \mathscr{F}(E)$ is l.s.c. and convex-valued because so is φ and W is open and convex in E. Define an usco mapping $\Psi_W : A_W \to \mathscr{C}(E)$ by $\Psi_W(x) = \theta(x) \cap H_W$, $x \in A_W$. By Michael's result [16], φ_W has an usco multi-selection $\Theta_W : M \to \mathscr{C}(E)$. Then, define another usco mapping $\theta_W : M \to \mathscr{C}(E)$ by $\theta_W(x) = \Psi_W(x) \cup \Theta_W(x)$ if $x \in A_W$ and $\theta_W(x) = \Theta_W(x)$ otherwise. Since θ_W is a multi-selection for φ_W, by Nepomnyashchii's result [21], φ_W admits a continuous multi-selection $\psi_W : M \to \mathscr{C}(E)$ such that $\theta_W(x) \subset \psi_W(x)$, $x \in M$. We may now define the required $\psi : M \to \mathscr{C}(E)$ by $\psi(x) = \bigcup\{\psi_W(x)\colon\ W \in \mathscr{W}\}$, $x \in M$. $\square$

In what follows, to every finite family $\mathscr{W}$ of subsets of Y we are going to associate the following subset of $\mathscr{C}(Y)$:

$$\langle \mathscr{W} \rangle = \Big\{S \in \mathscr{C}(Y)\colon\ S \subset \bigcup \mathscr{W} \text{ and } S \cap W \neq \varnothing, \text{ whenever } W \in \mathscr{W}\Big\}.$$

While we will not explicitly rely on this fact, let us mention that the collection of all such families $\langle \mathscr{W} \rangle$, where $\mathscr{W}$ runs over the finite families of open subsets of Y, is a base for the Vietoris topology on $\mathscr{C}(Y)$. It is well known that if (Y,d) is a metric space, then the Vietoris topology coincides with the Hausdorff topology on $\mathscr{C}(Y)$ generated by the Hausdorff distance associated to d.

Lemma 6.5. *Let Z be metrizable, E be a Banach space, $\varphi : Z \to \mathscr{F}(E)$ be an l.s.c. separable-convex-valued mapping, and let $\theta : Z \to \mathscr{C}(E)$ be an usco multi-selection for φ. Also, let $\mathscr{V}$ be an open cover of E consisting of convex sets, and let $\{U_V\colon\ V \in \mathscr{V}\}$ be a locally-finite open cover of E, with $\overline{U_V} \subset V$ for all $V \in \mathscr{V}$. Then, there exists a countable collection $\mathscr{S}(\mathscr{V})$ of continuous multi-selections $\psi : X \to \mathscr{C}(Y)$ for φ such that*

(i) *θ is a multi-selection of every $\psi \in \mathscr{S}(\mathscr{V})$,*
(ii) *for every finite $\mathscr{W} \subset \mathscr{V}$ and a point $z \in Z$, with $\theta(z) \in \langle\{U_W\colon\ W \in \mathscr{W}\}\rangle$, there is $\psi \in \mathscr{S}(\mathscr{V})$ such that $\psi(z) \subset \overline{\bigcup \mathscr{W}}$.*

Proof. We follow an idea in the proof of [5, Lemma 4.5]. Namely, take a base $\mathscr{B} = \bigcup\{\mathscr{B}_n\colon\ n < \omega\}$ for the topology of Z such that each family $\mathscr{B}_n$, $n < \omega$, is discrete and consists of closed sets. Also, let $[\mathscr{V}]^{<\omega}$ be the set of all nonempty finite subsets of $\mathscr{V}$. Consider the set

$$\mathscr{L} = \big\{(z,\sigma) \in Z \times [\mathscr{V}]^{<\omega}\colon\ \theta(z) \in \big\langle\{U_V\colon\ V \in \sigma\}\big\rangle\big\},$$

and let $\xi : \mathscr{L} \to Z$ and $\eta : \mathscr{L} \to [\mathscr{V}]^{<\omega}$ be the projections. Since θ is a multi-selection for φ, we have that $\varphi(z) \cap U_V \neq \varnothing$ whenever $\theta(z) \cap U_V \neq \varnothing$. Hence, we may define a map $\beta : \mathscr{L} \to \mathscr{B}$ such that, for every $\lambda \in \mathscr{L}$,

$$\xi(\lambda) \in \beta(\lambda) \subset \theta^{\#}\Big[\bigcup\big\{U_V\colon\ V \in \eta(\lambda)\big\}\Big] \cap \bigcap_{V \in \eta(\lambda)} \varphi^{-1}[U_V]. \tag{6.1}$$

Then, just like in the proof of [5, Lemma 4.5], we have that

$$\big|\eta\big(\beta^{-1}(B)\big)\big| \leqslant \omega \quad \text{for every } B \in \mathscr{B}. \tag{6.2}$$

Indeed, take $\lambda_0 \in \beta^{-1}(B)$. If $\lambda \in \beta^{-1}(B)$, then $\beta(\lambda) = B = \beta(\lambda_0)$, and, by (6.1), $\xi(\lambda_0) \in \beta(\lambda) \subset \bigcap_{V \in \eta(\lambda)} \varphi^{-1}[U_V]$. Consequently, $\varphi(\xi(\lambda_0)) \cap U_V \neq \varnothing$ for every $V \in \eta(\lambda)$, so $\eta(\lambda) \subset \{V \in \mathscr{V}\colon\ \varphi(\xi(\lambda_0)) \cap U_V \neq \varnothing\} = \mathscr{V}_0$. Since $\varphi(\xi(\lambda_0))$ is separable and $\{U_V\colon\ V \in \mathscr{V}\}$ is locally-finite, the family $\mathscr{V}_0$ must be countable. Thus, $[\mathscr{V}_0]^{<\omega}$ is also countable and (6.2) holds because $\eta(\lambda) \in [\mathscr{V}_0]^{<\omega}$.

We may now finish the proof as follows. For convenience, set

$$\mathscr{G}_n = \mathscr{B}_n \cap \beta(\mathscr{L}) = \big\{B \in \mathscr{B}_n\colon\ B = \beta(\lambda) \text{ for some } \lambda \in \mathscr{L}\big\}.$$

By (6.2), for every $B \in \mathscr{G}_n$, $n < \omega$, there is a surjective map $f_B : \omega \to \eta(\beta^{-1}(B))$. If $k < \omega$, then $f_B(k) \subset \mathscr{V}$ is a finite subset such that, by (6.1),

$$B \subset \theta^{\#}\Big[\bigcup\big\{U_V\colon\ V \in f_B(k)\big\}\Big] \cap \bigcap_{V \in f_B(k)} \varphi^{-1}[U_V].$$

So, by Proposition 6.4, $\varphi \restriction B$ has a continuous multi-selection $\Psi_{(B,k)} : B \to \mathscr{C}(E)$ such that, for every $x \in B$,

$$\theta(x) \subset \Psi_{(B,k)}(x) \subset \overline{\bigcup f_B(k)}. \tag{6.3}$$

However, $\mathscr{G}_n$ is a discrete family, and we may define another continuous mapping $\Psi_{(n,k)} : \bigcup \mathscr{G}_n \to \mathscr{C}(E)$ by $\Psi_{(n,k)} \restriction B = \Psi_{(B,k)}$ for every $B \in \mathscr{G}_n$. By (6.3), $\theta \restriction \bigcup \mathscr{G}_n$ is a multi-selection for $\Psi_{(n,k)}$. Then, by [10, Theorem 2.9] (see, also, [17, Theorem 1.2]), $\Psi_{(n,k)}$ can be extended to a continuous multi-selection $\psi_{(n,k)} : Z \to \mathscr{C}(E)$ for φ with $\theta(z) \subset \psi_{(n,k)}(z)$ for all $z \in Z$. According to (6.3) and the construction of the mapping $\Psi_{(n,k)}$, we have that $\theta(x) \subset \psi_{(n,k)}(x) = \Psi_{(B,k)}(x) \subset \overline{\bigcup f_B(k)}$ for every $x \in B \in \mathscr{G}_n$. Thus, the family $\mathscr{S}(\mathscr{V}) = \{\psi_{(n,k)}: n, k < \omega\}$ is as required. □

We conclude the preparation for the proof of Theorem 6.3 with the following consequence of Lemma 6.5.

Corollary 6.6. *Let Z be a metrizable space, E be a Banach space, and let $\theta : Z \to \mathscr{C}(E)$ be an usco mapping. Then θ is upper δ-continuous.*

Proof. Let $\varphi : Z \to \mathscr{C}(E)$ be an l.s.c. convex-valued mapping such that θ is a multi-selection for φ, see, for instance, Corollary 5.2. For every $n < \omega$, let $\mathscr{V}_n = \{y + 2^{-n}\mathbb{B}: y \in E\}$, where $\mathbb{B}$ is the unit open ball of E. Since E is metrizable, for each $n < \omega$ there exists an open locally-finite cover $\{U_V: V \in \mathscr{V}_n\}$ of E with $\overline{U_V} \subset V$ for all $V \in \mathscr{V}_n$. Then, by Lemma 6.5, for every $n < \omega$ there exists a countable family $\mathscr{S}_n$ of continuous mappings $\psi : Z \to \mathscr{C}(E)$ such that θ is a multi-selection for each $\psi \in \mathscr{S}_n$ and if $\mathscr{W} \subset \mathscr{V}_n$ is finite and $\theta(z) \in \langle\{U_W: W \in \mathscr{W}\}\rangle$ for some $z \in Z$, then $\psi(z) \subset \overline{\bigcup \mathscr{W}}$ for some $\psi \in \mathscr{S}_n$. According to the special choice of the cover $\mathscr{V}_n$, this actually means that $\psi(z) \subset \overline{\theta(z) + 2 \cdot 2^{-n}\mathbb{B}}$. Thus, the family $\mathscr{S} = \bigcup\{\mathscr{S}_n: n < \omega\}$ is as required. □

Proof of Theorem 6.3. The implication (a) ⇒ (b) is obvious, while (b) ⇒ (c) follows in exactly the same way as in the proof of (a) ⇒ (b) of [10, Lemma 4.7]. Finally, the implication (c) ⇒ (a) follows immediate by Corollary 6.6. The proof is completed. □

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